

Synthesis Method for the Spherical 4R Mechanism with Minimum Center of Mass Acceleration

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Abstract

In the mechanisms area, minimization of the magnitude of the acceleration of the center of mass (ACoM) implies shaking force balancing. This article shows an efficient optimum synthesis method for minimum acceleration of the center of mass of a spherical 4R mechanism by using dual functions, as well as, the counterweights balancing method. Once the dual function for ACoM has been formulated, one can minimize the shaking forces from a kinematic point of view. We present the synthesis of a spherical 4R mechanism for the case of a path generation task. The synthesis process involves the optimization of two objective functions, this multiobjective problem is solved by using the weighted sum method implemented in the evolutionary algorithm known as Differential Evolution. Our results shows that the magnitude of the ACoM can be reduced about 20 times. Which for the presented synthesis, is traduced in a reduction of about 89% for the shaking forces.

Keywords— Center of Mass Acceleration, Shaking Force, Spherical 4R Mechanism, Dual Numbers, Differential Evolution.

1 Introduction

A mechanism is said to be dynamically balanced if there is an elimination of dynamic reaction forces and torques (moments) on the base [1–4]. The use of counterweights [5–8] and springs [9–11] are the two most common methods for balancing a mechanism.

The balancing of mechanisms is a very active field in the mechanisms area, and there is a big amount of bibliography on such a topic, we refer the reader to works [12,13] where the balancing of mechanisms is reviewed. Specifically regarding to the concept of the center of mass, it is worthwhile to mention [14–16]. In [14], the balancing condition of parallel robots is obtained by expressing the position vector of the center of mass as a function of the position and orientation of the platform, and of six actuated prismatic joints. In [15], a synthesis method for linkages with the center of mass at an invariant link point is presented. In [16], the shaking force minimization of high-speed robots is done via optimal control of the acceleration of the total mass center of moving links.

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The problem of cancellation or minimization of the total external force can be addressed from a pure kinematic point of view by minimizing the ACoM. The idea of an optimization approach has been successfully applied in [17–21]. The smaller the ACoM, the smaller the shaking force in the mechanism. In the present work we show how, by using dual numbers [22–24] and the counterweights balancing method, a simple and computationally efficient optimum synthesis method with minimum center of mass acceleration can be achieved. First, we provide a method where the acceleration of the center of mass can be precisely and efficiently calculated. Second, since in some cases, a mass redistribution can become equal to zero the acceleration of the center of mass, we use an optimization based methodology, to find such a mass redistribution.

Our study is focused on the path generation task for a spherical 4R mechanism. The problem involves two objective functions to be optimized, namely the objective function to fulfill the desired trajectory and the objective function related to the acceleration of the center of mass. This multiobjective problem is solved by using the weighted sum method, which is implemented on the evolutionary algorithm known as differential evolution (DE) [25]. The synthesized mechanism was constructed, and actuated by a dc motor controlled by a classical PID law. This allow us to compare the required power (and energy used) to operate the balanced and unbalanced mechanisms. The experimental measurement of the energy consumption was contrasted with its theoretical calculation which is conducted by using the work–energy theorem, providing in this way an experimental validation for the reduction of the ACoM in the balanced mechanism.

The rest of the paper is organized as follows, section 2 presents the essentials of our method. There, the balancing method, the dual number approach to obtain derivatives and the optimization method are presented. In section 3 a Spherical 4R mechanism is synthesized for a prescribed path generation task with minimum center of mass acceleration. This section shows all the parameters and formulas that need to be written in its dual form in order to obtain the acceleration of the center of mass vector. Also a theoretical calculation of the energy consumption is presented. Section 4 shows the results along some discussions. Finally, the conclusions are presented in section 6.

2 Synthesis Method

In general, a mechanism has an arbitrary time-varying ACoM. So our aim is to minimize the magnitude of the ACoM. For that purpose we need a precise and efficient method to obtain first and second derivatives, as well as an optimization method that allow us to cancel or minimize the ACoM. This section shows the implementation of the dual numbers to obtain derivatives and the method used for balancing.

2.1 Minimum Acceleration of Center of Mass and Counterweight Method

There are several methods for balancing mechanisms, each one having advantages and disadvantages [3, 12, 13]. Since a mathematical point of view, more free parameters are introduced to the synthesis problem to balance a mechanism. We implement this approach for balancing a spherical 4R (Fig. 1) mechanism by adding counterweights [5–7, 14] to the links as shown in Fig. 2, therefore we have more independent variables in order to minimize the ACoM of the Spherical 4R mechanism.

2.2 Dual numbers approach

Usually, the process of calculating a derivative is not difficult. However, for the case of a spherical mechanism, obtaining first and second derivatives of the position vector for the center of mass is not simple. Even when such derivatives can be explicitly obtained, the resulting expressions could be of great complexity and useless for practical purposes. An alternative solution is to numerically calculate such derivatives. Nevertheless, traditional methods for calculating numerical derivatives (finite-differences) are not efficient enough to be used in the optimum synthesis of mechanisms. Moreover, they are subject to both truncation and subtractive cancellation errors. This is an issue to be dealt with in the optimum synthesis of mechanisms, since in some cases the errors introduced by the numerical derivative method are of the same order than the quantity to be optimized.

A different approach that is not subject to the above mentioned errors can be constructed by using dual numbers [26–28]. In order to make this paper self contained we briefly review the essential ideas following [28] and bearing in mind a numerical implementation.

Dual numbers and derivatives

A dual number \hat{r} is a number of the form

$$\hat{r} = a + \epsilon b, \quad (1)$$

where a (the real coefficient) and b (the dual coefficient) are real numbers and $\epsilon^2 = 0$. As in the case of complex numbers, a dual number can also be defined as ordered pairs

$$\hat{r} = [a, b]. \quad (2)$$

The algebraic rules for dual numbers can be found elsewhere in the literature, see for example [22–24].

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which accepts a Taylor expansion we have:

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \dots + O(h^3). \quad (3)$$

Now, considering the particular dual number $\hat{x} = x + \epsilon$, i.e., a dual number where the coefficient of the nilpotent ϵ is equal to one we have $\hat{f}(\hat{x}) = f(x) + f'(x)\epsilon$. In the notation of (2) we may write

$$\hat{f}(\hat{x}) = [f(x), f'(x)]. \quad (4)$$

So, if instead of calculating over the reals, we calculate over the *duals*, we end up with a dual function where the real part is the function itself and the dual part its derivative.

The extension to obtain the second derivative is straightforward. For this purpose we promote the $f'(x)$ function to be a dual function, that is:

$$\hat{f}'(\hat{x}) = [f'(x), f''(x)]. \quad (5)$$

The information of the function itself and its first and second derivative can be stored in a vector of three components. We will use the notation

$$\tilde{f}(x) = [f(x), f'(x), f''(x)] \quad (6)$$

to represent the extended dual version of the original function f .

The following pseudo-code implements the extended dual version $\tilde{f} = [f_0, f_1, f_2]$ of the function f , as a function of the dual variable $\tilde{g} = [g_0, g_1, g_2]$:

```

start the definition of the dual function
declare the type of variables (if necessary)
dualf = [f_0(g_0), f_1(g_0)*g_1, f_2(g_0)*g_1^2 + f_1(g_0)*g_2 ]
end the definition of the dual function.

```

Following the above discussion, one can obtain the extended dual version of the center of mass vector from where its velocity and acceleration are directly obtained. The derivatives are calculated to the same precision of the implemented standard functions of the used programming language, and its calculation is reduced to a simple function evaluation without the need of a limit calculation process. As an added value to our work, some useful functions common in the rotational kinematics are dualized for the first time and provided as additional material to this article¹. So they will be of great help to any one interested in applying the dual numbers to obtain derivatives. Since we are interested in the optimal synthesis of mechanisms (fast and efficient programs are desired), we have coded the dual functions in Fortran but a translation to a more user-friendly language is almost direct.

2.3 Optimization method

The optimal dimensional synthesis is conducted by using the evolutionary DE method [25]. This method has been successfully applied to the optimal dimensional synthesis of mechanisms [29–32] and mainly consists of three operators: mutation, crossover and selection; applied to a possible solution or individual \mathbf{x} . Below an outline of the DE method, following [25] is presented.

2.3.1 Differential Evolution Algorithm

1. The population is described by:

$$\begin{aligned}
 \mathbf{P}_{\mathbf{x},g} &= (\mathbf{x}_{i,g}), \quad i = 1, \dots, m; \quad g = 0, \dots, g_{\max} \\
 \mathbf{x}_{i,g} &= (x_{i,g}^j), \quad j = 1, \dots, D;
 \end{aligned} \tag{7}$$

where D represents the dimension of \mathbf{x} , m represents the number of individuals, and g is the generation.

2. Initialization of population.

$$x_{i;0}^j = rand^j(0, 1) \cdot (b_U^j - b_L^j) + b_L^j.$$

Vectors \mathbf{b}_U and \mathbf{b}_L are the limits of parameters. $rand^j(0, 1)$ represents uniformly distributed random number in $[0, 1)$. Superscript j means that a random value for each parameter is generated.

3. Mutation.

$$\mathbf{v}_{i;g} = \mathbf{x}_{r_0;g} + F \cdot (\mathbf{x}_{r_1;g} - \mathbf{x}_{r_2;g}). \tag{8}$$

The main difference between DE and other evolutionary algorithms like genetic algorithms is due to this mutation operator. $\mathbf{x}_{r_0;g}$ is called the base vector which is perturbed by the difference of other two vectors. $r_0, r_1, r_2 \in \{1, 2, \dots, m\}$, $r_1 \neq r_2 \neq r_3 \neq i$. F , the mutation scale factor, is a real number greater than zero.

¹A file (Dual.ACoM.files.zip) containing the code of such functions can be downloaded from <http://www.meca.cinvestav.mx/personal/cacruz/archivos-ccv/>

4. Crossover.

It uses a dual recombination of vectors to generate the trial vector:

$$\mathbf{u}_{i;g} = u_{i;g}^j = \begin{cases} v_{i;g}^j & \text{if } \text{rand}^j(0, 1) \leq Cr \text{ or } j = j_{\text{rand}} \\ x_{i;g}^j & \text{otherwise.} \end{cases} \quad (9)$$

The crossover probability, $Cr \in [0, 1]$, is a user-defined value, $j_{\text{rand}} \in [1, D]$.

5. Selection.

The selection is made according to

$$\mathbf{x}_{i;g+1} = \begin{cases} \mathbf{u}_{i;g} & \text{if } f(\mathbf{u}_{i;g}) \leq f(\mathbf{x}_{i;g}) \\ \mathbf{x}_{i;g} & \text{otherwise} \end{cases} \quad (10)$$

2.3.2 Objective function for the prescribed path generation task

In this study we are interested in the prescribed path generation task. So, besides an objective function related to the minimization of the ACoM, we also consider an objective function intended to fulfill the path generation task.

In the path generation problem, n points are given and a mechanism passing through these desired points is required. In order to synthesize such a mechanism (in this study a spherical 4R mechanism), we minimize the mean structural error (root mean square error) which is defined as:

$$f_{\text{path}}(\theta_1, \phi_1, \eta_1, \psi, \beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{n} \sum_{i=1}^n \|\mathbf{r}_{di} - \mathbf{r}_{\text{gen}i}\|^2 \right)^{1/2}, \quad (11)$$

where $\mathbf{r}_{di} - \mathbf{r}_{\text{gen}i}$ is the difference between the i -th desired point \mathbf{r}_{di} and the i -th generated point $\mathbf{r}_{\text{gen}i}$. The θ_1 parameter, is the angle for the input link corresponding to the first point. In the prescribed path generation task the angle corresponding to the other points are given.

2.3.3 Objective function for minimum center of mass acceleration

In order to obtain a mechanism with minimum ACoM we minimize the objective function

$$f_{\text{cm}}(\theta_1, \phi_1, \eta_1, \psi, \beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4, e_1, e_2, e_3, e_4) = \left(\frac{1}{n} \sum_{i=1}^n \|\ddot{\mathbf{r}}_{\text{cm};i}\|^2 \right)^{1/2} \quad (12)$$

where $\ddot{\mathbf{r}}_{\text{cm};i}$ is the acceleration of the center of mass of the mechanism when it is passing through the i -th point.

3 Synthesis of a Spherical 4R Mechanism

This section presents the application of the method above explained, to the synthesis of a spherical 4R mechanism for a path generation task, with minimum ACoM. Table 1 (as presented in [33]) shows the desired points. This problem was originally studied in [34] but there, the desired points are not on a unit sphere.

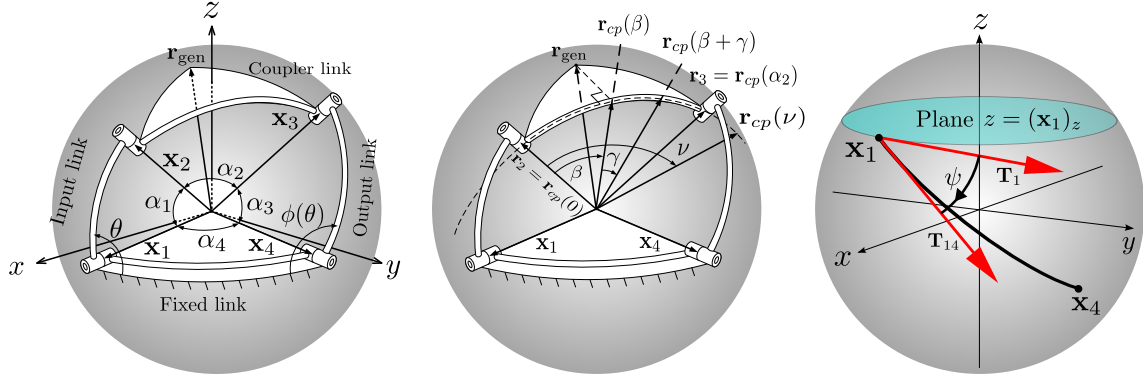


Figure 1: Spherical 4R Mechanism and variables used to obtain \mathbf{r}_{gen} .

3.1 Kinematics

A spherical four-bar mechanism (Fig. 1) is a closed chain, consisting of four links and four revolute joints. These linkages have the property that every link in the system rotates about the same fixed point [35]. Thus, in a spherical mechanism, any point in a moving body is confined to move within a spherical surface, and all spherical surfaces of motion are concentric. In order to find the coupler point curve \mathbf{r}_{gen} , we will use the parameters shown in Fig. 1. Such equation has been already obtained in [36], however, we want to change to a more concise description by using the lengths of the links as parameters instead of the coordinates for the joints. Thus, the coupler-point curve can be written as:

$$\mathbf{r}_{\text{gen}}(\theta, \phi_1, \eta_1, \psi, \beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbf{R}(\pi/2, \mathbf{r}_{cp}(\beta)) \mathbf{r}_{cp}(\beta + \gamma), \quad (13)$$

where ϕ_1 and η_1 are the azimuth and polar angle for the point \mathbf{x}_1 , $\mathbf{R}(\nu, \hat{\mathbf{w}})$ is the active rotation matrix of angle ν in direction of the unit vector $\hat{\mathbf{w}}$.

$$\mathbf{r}_{cp}(\nu) = \mathbf{R}(\nu, \hat{\mathbf{n}}_{23}) \mathbf{r}_2(\theta, \psi, \phi_1, \eta_1, \alpha_1, \alpha_4), \quad (14)$$

$$\hat{\mathbf{n}}_{23} = \frac{\mathbf{r}_2(\theta, \psi, \phi_1, \eta_1, \alpha_1, \alpha_4) \times \mathbf{r}_3(\phi(\theta), \psi, \phi_1, \eta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4)}{\|\mathbf{r}_2(\theta, \psi, \phi_1, \eta_1, \alpha_1, \alpha_4) \times \mathbf{r}_3(\phi(\theta), \psi, \phi_1, \eta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4)\|}, \quad (15)$$

$$\mathbf{r}_2(\theta, \psi, \phi_1, \eta_1, \alpha_1, \alpha_4) = \mathbf{R}(\theta, \mathbf{x}_1) \mathbf{R}(\alpha_1, \hat{\mathbf{n}}_{14}) \mathbf{x}_1. \quad (16)$$

For $\eta_1 \neq 0, \pi$, we obtain²

$$\hat{\mathbf{n}}_{14} = \frac{\mathbf{x}_1 \times \mathbf{T}_{14}}{\|\mathbf{x}_1 \times \mathbf{T}_{14}\|} \quad (17)$$

$$\mathbf{T}_{14} = \mathbf{R}(\psi, \mathbf{x}_1) \mathbf{T}_1 \quad (18)$$

$$\mathbf{T}_1 = \left. \frac{\partial}{\partial u} \left(\mathbf{R}(u, \hat{\mathbf{k}}) \mathbf{x}_1 \right) \right|_{u=0} \quad (19)$$

²Similar formulas can be obtained when $\eta_1 = 0$ or $\eta_1 = \pi$ although a simple solution is to chose another basis, for instance $\{\hat{\mathbf{k}}, \hat{\mathbf{i}}, \hat{\mathbf{j}}\}$.

Table 1: Desired points for the prescribed timing path generation task.

Point number	Point	Point number	Point
1	(0.85737, -0.18481, 0.48037)	33	(0.7887, -0.60370, 0.11578)
2	(0.82985, -0.20167, 0.52030)	34	(0.80152, -0.59270, 0.07900)
3	(0.80241, -0.21996, 0.55478)	35	(0.81378, -0.57959, 0.04311)
4	(0.77567, -0.23967, 0.58389)	36	(0.82552, -0.56433, 0.00841)
5	(0.75011, -0.26056, 0.60785)	37	(0.83678, -0.54700, -0.02478)
6	(0.72607, -0.28244, 0.62693)	38	(0.84759, -0.52763, -0.05611)
7	(0.70381, -0.30515, 0.64152)	39	(0.85807, -0.50641, -0.08530)
8	(0.68352, -0.32833, 0.65193)	40	(0.86819, -0.48344, -0.11200)
9	(0.66533, -0.35185, 0.65844)	41	(0.87804, -0.45889, -0.13596)
10	(0.64933, -0.37537, 0.66144)	42	(0.88763, -0.43304, -0.15689)
11	(0.63559, -0.39867, 0.66115)	43	(0.89704, -0.40611, -0.17448)
12	(0.62415, -0.42159, 0.65781)	44	(0.90626, -0.37837, -0.18848)
13	(0.61504, -0.44389, 0.65167)	45	(0.91537, -0.35022, -0.19867)
14	(0.60833, -0.46541, 0.64293)	46	(0.92433, -0.32193, -0.20481)
15	(0.60396, -0.48596, 0.63170)	47	(0.93322, -0.29396, -0.20667)
16	(0.60196, -0.50548, 0.61819)	48	(0.94196, -0.26667, -0.20400)
17	(0.60230, -0.52381, 0.60241)	49	(0.95052, -0.24048, -0.19667)
18	(0.60485, -0.54085, 0.58448)	50	(0.95885, -0.21581, -0.18441)
19	(0.60959, -0.55656, 0.56448)	51	(0.96685, -0.19315, -0.16707)
20	(0.61637, -0.57081, 0.54244)	52	(0.97430, -0.17289, -0.14452)
21	(0.62500, -0.58363, 0.51844)	52	(0.98096, -0.15541, -0.11656)
22	(0.63530, -0.59489, 0.49248)	54	(0.98652, -0.14104, -0.08315)
23	(0.64700, -0.60456, 0.46467)	55	(0.99052, -0.13007, -0.04430)
24	(0.65989, -0.61259, 0.43507)	56	(0.99244, -0.12263, -0.00019)
25	(0.67370, -0.61896, 0.40378)	57	(0.99174, -0.11870, 0.04874)
26	(0.68811, -0.62363, 0.37093)	58	(0.98774, -0.11822, 0.10185)
27	(0.70293, -0.62652, 0.33674)	59	(0.98000, -0.12085, 0.15807)
28	(0.71789, -0.62759, 0.30133)	60	(0.96819, -0.12626, 0.21604)
29	(0.73274, -0.62678, 0.26500)	61	(0.95226, -0.13415, 0.27426)
30	(0.74737, -0.62404, 0.22800)	62	(0.93252, -0.14411, 0.33115)
31	(0.76167, -0.61933, 0.19059)	63	(0.90956, -0.15600, 0.38515)
32	(0.77544, -0.61256, 0.15307)	64	(0.88422, -0.16959, 0.43519)

$$\mathbf{r}_3(\phi(\theta), \psi, \phi_1, \eta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbf{R}(\phi(\theta), -\mathbf{x}_4) \mathbf{R}(\alpha_3, -\hat{\mathbf{n}}_{14}) \mathbf{x}_4, \quad (20)$$

$$\mathbf{x}_4 = \mathbf{R}(\alpha_4, \hat{\mathbf{n}}_{14}) \mathbf{x}_1. \quad (21)$$

Finally, the output angle $\phi(\theta)$ is given by:

$$\phi(\theta) = 2 \tan^{-1} \left(\frac{-A \pm \sqrt{A^2 + B^2 - C^2}}{C - B} \right) \text{ where} \quad (22)$$

$$A = \sin \alpha_1 \sin \alpha_3 \sin \theta$$

$$B = \cos \alpha_1 \sin \alpha_3 \sin \alpha_4 - \sin \alpha_1 \sin \alpha_3 \cos \alpha_4 \cos \theta$$

$$C = \cos \alpha_1 \cos \alpha_3 \cos \alpha_4 + \sin \alpha_1 \cos \alpha_3 \sin \alpha_4 \cos \theta - \cos \alpha_2.$$

It is worthwhile to mention that, since points \mathbf{x}_2 and \mathbf{x}_3 are not given as input parameters, we arbitrarily can choose any of the two branches. Notice however, that if points \mathbf{x}_2 and \mathbf{x}_3 are given

as input parameters, then a careful choice of the branch is needed (see for example appendix C of [37] for details).

3.2 Center of Mass Acceleration

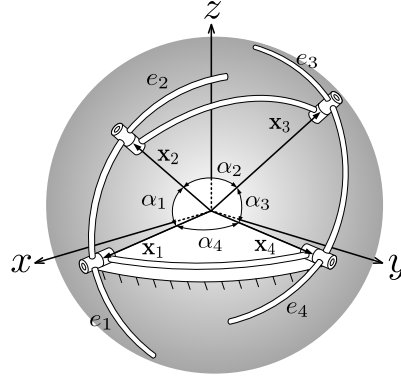


Figure 2: Extensions of the Input and Output Links.

Let us consider that a link has the geometry of a geodesic joining two points, \mathbf{p}_1 and \mathbf{p}_2 on the unit sphere, whose associated vectors subtends an arc α . Assuming that the link is made of an homogeneous material, the center of mass vector of the link is

$$\mathbf{r}_{cm;l} = \frac{2}{\alpha} \sin(\alpha/2) \mathbf{R}(\alpha/2, \hat{\mathbf{p}}_{12}) \mathbf{p}_1, \quad (23)$$

where $\hat{\mathbf{p}}_{12} = \mathbf{p}_1 \times \mathbf{p}_2 / \|\mathbf{p}_1 \times \mathbf{p}_2\|$.

Since the density of mass is a constant, the center of mass vector of the input link (without extensions) can be calculated as:

$$\mathbf{r}_{cm;l_1} = \frac{2}{\alpha_1} \sin(\alpha_1/2) \mathbf{R}(\alpha_1/2, \hat{\mathbf{n}}_{12}) \mathbf{x}_1 \quad (24)$$

with

$$\hat{\mathbf{n}}_{12} = \frac{\mathbf{x}_1 \times \mathbf{r}_2(\theta, \psi, \phi_1, \eta_1, \alpha_1, \alpha_4)}{\|\mathbf{x}_1 \times \mathbf{r}_2(\theta, \psi, \phi_1, \eta_1, \alpha_1, \alpha_4)\|}. \quad (25)$$

The center of mass for the other links and extensions (Fig. 2) can be similarly calculated. Once all the center of mass for the links and extensions have been calculated, the center of mass of the complete mechanism is:

$$\mathbf{r}_{cm} = \sum_k s_k \mathbf{r}_{cm;k} / \sum_k s_k, \quad (26)$$

where s_k is the length of the k -th link (or extension) for $k = e_1, e_2, e_3, e_4, l_1$, etc.

Now, the acceleration of the center of mass will be

$$\ddot{\mathbf{r}}_{cm} = \ddot{\theta} \frac{\partial \mathbf{r}_{cm}}{\partial \theta} + \dot{\theta}^2 \frac{\partial^2 \mathbf{r}_{cm}}{\partial \theta^2}. \quad (27)$$

Since it is possible to control the angular velocity of the input link, the calculation of the acceleration of the center of mass is reduced to calculate $\partial \mathbf{r}_{cm} / \partial \theta$ and $\partial^2 \mathbf{r}_{cm} / \partial \theta^2$. Both first and second

derivatives are obtained when we write Eq. (26) in its dual form (see the `rcm_dual_mod.f90` file of the additional material). In order to clarify, let us find the dual version for the center of mass of the input link. Let $\tilde{\mathbf{R}}(\tilde{\theta}, \tilde{\mathbf{u}})$ be the dual version of the active rotation matrix of dual angle $\tilde{\theta}$ about the unit dual vector $\tilde{\mathbf{u}}$. Then

$$\tilde{\mathbf{r}}_{cm;l_1} = \frac{2}{\alpha_1} \sin(\alpha_1/2) \tilde{\mathbf{R}}(\tilde{\alpha}_1/2, \tilde{\mathbf{n}}_{12}) \tilde{\odot} \tilde{\mathbf{x}}_1 \quad (28)$$

where $\tilde{\odot}$ represents the dual multiplication of a dual matrix with a dual vector, $\tilde{\alpha}_1 = [\alpha_1, 0, 0]$, $\tilde{\mathbf{n}}_{12}$ the dual version of Eq. (25), which, in order to be constructed all the involved operations need to be promoted to dual operations. Needless to say, such operations are coded in the downloadable module. It is interesting to note that neither the sum nor the scalar multiplication, need to be dualized, this is, of course due to the linearity of the derivative operator.

3.3 Theoretical calculation of the energy consumption.

From the theoretical point of view, the energy consumption of the spherical 4R mechanism can be calculated as follows.

The external forces to the spherical 4R mechanism are,

- \mathbf{R}_1 : The reaction force of the frame on the \mathbf{x}_1 point.
- \mathbf{R}_2 : The reaction force of the frame on the \mathbf{x}_4 point.
- \mathbf{F}_g : The weight of the mechanism.
- \mathbf{F}_M : The external force due to the motor (the agent who generates the rotation of the input link).

In our description, the work done by \mathbf{R}_1 and \mathbf{R}_2 is zero, as the points \mathbf{x}_1 and \mathbf{x}_4 are fixed. Since \mathbf{F}_g is a conservative force, the work done by this force will be $W_{\mathbf{F}_g} = -\Delta U$, where, choosing the acceleration of gravity as $\mathbf{g} = -g\hat{\mathbf{k}}$, the gravitational potential energy for the mechanism of mass m is,

$$U(\theta) = m g z_{cm}(\theta), \quad (29)$$

with $z_{cm}(\theta)$ the z -coordinate of the center of mass as a function of the input angle θ .

From the work-energy theorem, a differential amount of work done by the force \mathbf{F}_M can be written as

$$\delta W_{\mathbf{F}_M} = (dU + dK) + \delta E_{\text{others}}, \quad (30)$$

where K is the kinetic energy and the positive quantity δE_{others} represents all the other kinds of energies we have not considered, for instance the heat dissipated by friction, etc.

The energy expended by the external agent (the motor) will be

$$\delta E = |(dU + dK)| + \delta E_{\text{others}}. \quad (31)$$

Notice that the absolute value is used, this is because the quantity $(dU + dK)$ can be negative and it could happen that $\delta W_{\mathbf{F}_M}$ to be zero³, which in fact could be true. However, this fact does

³From a theoretical point of view, if the quantity δE_{others} is known, one can obtain an angular velocity profile –eigenmotion– for the input link in such a way that $\delta W_{\mathbf{F}_M} = 0$.

not mean that the motor moves the mechanism without consuming energy. In our case, with a conventional motor operating the mechanism, there is no difference between a negative work and a positive work since with a conventional motor we cannot reuse the energy taken from the mechanism. In the best scenario the energy consumption will be δE_{others} .

Considering U and K as functions of θ , using a Taylor expansion we have

$$U(\theta + d\theta) - U(\theta) = U'(\theta)d\theta \quad (32)$$

from where

$$dU = U'(\theta)d\theta \quad (33)$$

and similarly

$$dK = K'(\theta)d\theta. \quad (34)$$

Thus in a cycle the energy consumption is given by

$$E = \int_0^{2\pi} |U'(\theta) + K'(\theta)|d\theta + E_{\text{others}}. \quad (35)$$

Applying this equation to the balanced (B) and unbalanced (NB) mechanisms, we have,

$$E_{NB} - E_B = \int_0^{2\pi} (|U'(\theta) + K'(\theta)|_{NB} - |U'(\theta) + K'(\theta)|_B) d\theta + (E_{\text{others};NB} - E_{\text{others};B}). \quad (36)$$

A quantification of E_{others} is not a trivial task. However if both mechanisms are constructed as similar as possible (except of course by the extensions), the frictional forces are also as reduced as possible and if the deformation energy for both mechanisms are either similar or small, then a plausible approximation is that $E_{\text{others};NB} \approx E_{\text{others};B}$. Of course this approximation is not in general true, but in our case it can be justified *a posteriori* as our theoretical results are in good agreement with the experimental measure for $E_{NB} - E_B$.

4 Results

4.1 Optimum synthesis with minimum acceleration of the center of mass

The multiobjective optimization problem (simultaneous optimization of Eqs. (11) and (12)) was solved by using the weighted sum method and the evolutionary algorithm known as Differential Evolution [25]. Specifically, we use the DE/rand/1/bin strategy with the dither variant. For the numerical results we have used a population of 50 individuals and a number of generations of 10 000 and a crossover probability of 0.9.

Assuming a constant angular velocity of 1 rad/s for the input link (we use a PID controller to regulate this velocity), the DE method gives as result the following values: $\mathbf{x}_1 = [1.0000, 0.0000, 0.0000]$, $\mathbf{x}_4 = [0.54462, 0.80817, 0.22413]$, $\alpha_1 = 0.40144$ rad, $\alpha_2 = 0.82035$ rad, $\alpha_3 = 0.92505$ rad, $\beta = 0.23066$ rad, $\gamma = 0.47437$ rad, $e_1 = 4.46818$ rad, $e_2 = 0.94312$ rad, $e_3 = 0.24879$ rad and $e_4 = 4.30884$ rad. With these design parameters we obtain $f_{\text{path}} = 0.00002$ m and $f_{\text{cm}} = 0.01351$ m/s². Unfortunately it is not possible to build this mechanism, as the links (with extensions) would collide with the supports for the frame points (\mathbf{x}_1 and \mathbf{x}_4). A feasible mechanism can be synthesized demanding that the z -coordinate for the points \mathbf{x}_1 and \mathbf{x}_4 to be zero,

Table 2: Parameters of the constructed spherical 4R mechanism.

\mathbf{x}_1			\mathbf{x}_4			α_1	α_2	α_3	β	γ	e_1	e_2	e_3	e_4
1.00	0.00	0.00	0.39	0.92	0.00	0.40	0.88	1.39	0.34	0.40	1.92	1.06	0.00	2.79



Figure 3: Constructed spherical 4R mechanism.

and choosing horizontal supports for the frame points, thus avoiding collisions with the extensions. With these considerations, we obtain the mechanism whose parameters are shown in Table 2. Fig. 3 shows a first prototype of the constructed mechanism. For this mechanism $f_{path} = 0.00053$ m, while $f_{cm} = 0.014$ m/s². It is worthwhile to note that if no extensions are considered for the mechanism; i.e. $e_i = 0$ for $i = 1, \dots, 4$; the multiobjective optimization method yields $f_{cm} = 0.28$ m/s² which is twenty times bigger than the corresponding value for the mechanism with extensions. As a result, the shaking forces are minimized. From Fig. 4 we can see that the reduction of the shaking force amplitude for the balanced mechanism, reaches values of 95% and in the worst scenario of 78%, having a mean value of 89%. Notice that this reduction is independent (from a theoretical point of view) of the used angular velocity for the input link ($\dot{\theta}$). Because if $\dot{\theta}$ in Eq. (27) is constant, the ratio between ACoM for the balanced and unbalanced mechanisms, does only depend on the ratio of their second partial derivatives with respect to θ .

4.2 Experimental validation

In order to validate our theoretical results (shown in Fig 4), either a measurement of the ACoM or of the external forces is required. Such experimental measurements are not easy to be performed. Fortunately an experimental validation of theoretical results can be made by measuring the difference in the energy consumption $E_{NB} - E_B$, of the balanced and unbalanced mechanisms. The fact that the quantity under the integral sign in Eq. (36) depends on the velocity and the acceleration of the center of mass allows to compare such theoretical quantity with its experimental counterpart. To this end, we have manufactured both mechanisms with and without extensions, considering a sphere of mean radius⁴ equal to 0.23 m. Specifically, the radii for the input link the coupler link and the output links were of 0.250 m, 0.235 m and 0.210 m respectively. The radius for the links forming the triangle (Fig. 3) for the coupler point was of 0.255 m. The links were

⁴The links of the spherical 4R mechanism were constructed in concentric spheres of different radius. The deviation in the radius for the links depends on the thickness of the links, in our case 6.35 mm.

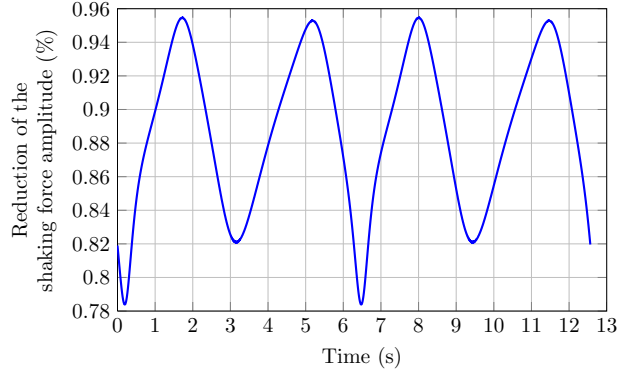


Figure 4: Reduction of the shaking force amplitude for the mechanism with minimum center of mass acceleration. The mean value is of 89%.

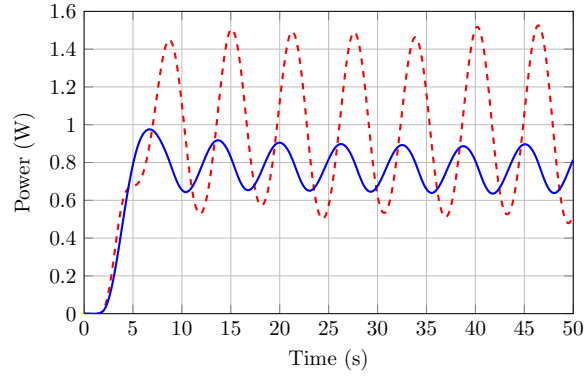


Figure 5: Experimental power consumption comparison between the ACoM balanced (solid line) and unbalanced (dashed line) mechanisms.

constructed with aluminum slab of 6.35 mm thickness and 2.54 cm width. For both mechanisms a dc motor is used as actuator to the input link, which is controlled by a PID law with a reference of 1 rad/s. The energy required to operate the mechanism is considered equal to the electrical energy required to actuate the motor, which was obtained from direct measurement of voltage and current at its terminals. In Fig 5, we show the power consumption for both mechanisms. Fig 6, presents the time behavior of the angular velocity of the input link of both mechanisms (which is obtained as the output of the filter $s/(1 + 0.01s)$, whose input comes from a digital encoder attached to the joint of the input link), we can observe that the regulation objective is not absolutely reached, however the oscillations around the reference are smaller for the case of the balanced mechanism.

After an integration of the experimental data showed in Fig. 5, we have obtained a deviation of only 5% with respect to a theoretical calculation of $E_{NB} - E_B$ using Eq. (36) and neglecting $E_{others;NB} - E_{others;B}$.

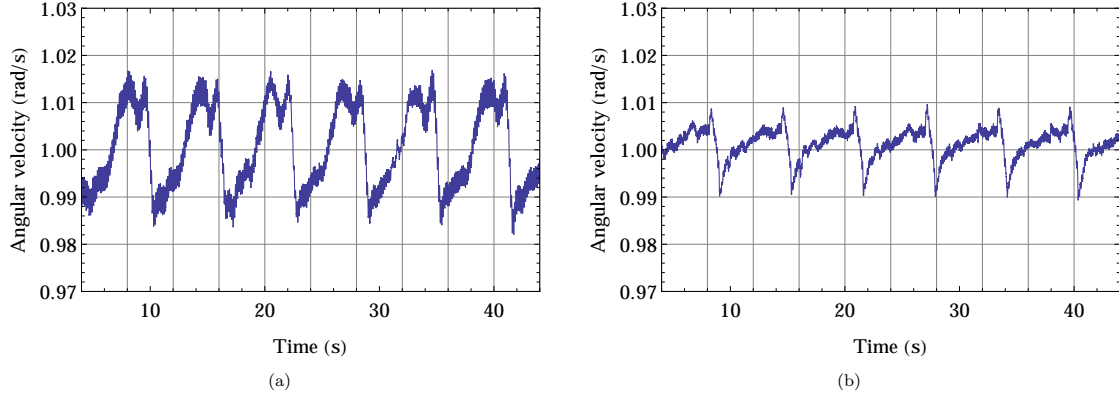


Figure 6: Controlled velocity for the unbalanced mechanism (a) and the ACoM balanced mechanism (b).

5 Discussion

Before concluding, some comments on the used derivative method and the energy consumption are worthwhile. Below we present a brief discussion.

Numerical derivatives: Dual numbers vs Finite Differences

Using finite differences (FD), the second derivative of a function can be calculated as,

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \quad (37)$$

In order to obtain the second derivative using this expression, three evaluations of the function f are required. Since in the dual number approach we only need one evaluation, it is reasonable to think that this method will be more efficient. However, we cannot say that the dual number approach will be three times faster than the finite difference method, as the involved evaluations do not correspond to the same functions. In our case, for 5000 points equally spaced in $(0, 2\pi)$, the matrix $[\dot{\mathbf{r}}_{cm}, \ddot{\mathbf{r}}_{cm}]$ was calculated 2.5 times faster by using the dual numbers approach instead of the FD method.

Now, from Eq. (37) a truncation error of order h^2 is present. One could think that this problem can be solved by taking a small value for h , but then subtractive cancellation errors will appear. For example taking $h = 1 \times 10^{-6}$ and using double precision for the variables, the error in the calculation of the ACoM is not of the order h^2 but of the order of 10^{-2} . This situation can be alleviated by writing a more elaborated formula for f'' but there is a cost to pay, we will need more evaluations of the function f . These problems are not present when the dual numbers approach is used to calculate the derivatives. Moreover, this fact is not the only advantage of the dual number approach. For instance, one could be interested in obtaining the derivative of a function that is the solution of some equation which can only be known numerically. In this case, we can implement the numerical solution method in the context of the dual number approach and then both, the solution and its derivative will be directly obtained.

Energy consumption

Even when minimization of ACoM does not imply a constant gravitational potential energy – excepting if ACoM is zero and for a mechanism operating in cycles⁵ –, we have found (integrating the absolute value of Eq. (33)) that the variations in the potential energy are reduced about 92% for the balanced mechanism.

It is worthwhile to mention that not only the shaking forces were reduced by balancing the mechanism but also the energy consumption (obtained by integrating the experimental data of Fig. 5) was reduced by a 21 %. Of course this is not a rule since we are not optimizing Eq. (35). Moreover, unlike the percentile reduction of the total external force, the percentile reduction of the energy consumption depends on the angular velocity of the input link.

6 Conclusions

By implementing the center of mass vector in its dual version, a simple yet effective method for balancing a spherical 4R mechanism using the counterweights method has been presented. The minimization of the magnitude of the center of mass acceleration vector has been conducted with the Differential Evolution algorithm. The synthesis process for the prescribed path generation task turn out to be highly efficient giving excellent results. The reduction of the force shaking amplitude, has reached a mean value of 89% while the variations in the gravitational potential energy have been reduced about 92%. The synthesized mechanisms, the balanced and its unbalanced counterpart, were constructed in order to experimentally measure the power consumption and hence the energy needed for both mechanisms to perform the requested path. By doing this, an experimental validation of theoretical results has been conducted. As a side effect, it is interesting to note that the energy used to operate the balanced mechanism was reduced by 21% with respect to its unbalanced counterpart. Nevertheless this cannot be considered as a rule as a minimization of ACoM does not necessarily implies a minimization of Eq (35).

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⁵Let us assume that ACoM is zero, this means that the velocity of the center of mass is a constant. Then, in a cycle the mechanism must return to its initial point, therefore the only allowed constant for the velocity of the center of mass is zero. Thus GPE is constant. Notice that the converse is not true.

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